PHYSICAL REVIEW VOLUME 132, NUMBER 2 15 OCTOBER 1963

Nuclear Size Effect on Bremsstrahlung in the BeV Range*

B. J. MALENKA AND H. P. SMITH
Department of Physics, Northeastern University, Boston, Massachusetts
(Received 18 March 1963; revised manuscript received 20 June 1963)

Since a more detailed knowledge of the bremsstrahlung cross section at high energies may be required for certain photoproduction experiments, the percentage correction to the Bethe-Heitler point-nucleus results arising from the finite size of the nuclear-charge distribution is estimated. The effect of a shape-independent approximation to the form factor is considered. The results presented in Table I of the text indicate corrections of less than a few percent for incident electrons wtih energies in the BeV range. A comparison with previous work is also made.

I. INTRODUCTION

As the new multi-BeV electron accelerators are brought into full operation, the bremsstrahlung beams obtained from these electrons will be used in various photoproduction and other types of experiments. In order to do such experiments accurately, it is desirable to have as detailed and precise a knowledge of the bremsstrahlung cross sections as is possible. This is especially so for measuring photoproduction by a difference experiment, where the shape of the bremsstrahlung spectrum in the high-frequency region is of particular importance. It was in the course of investigating this problem, that the question of the effect of finite nuclear size was raised.

As is well known, electrons in the BeV range have a de Broglie wavelength that is small compared to nuclear radii. Thus, such electrons were expected to probe some of the details of nuclear structure. These expectations have been well born out by the high-energy electron-scattering experiments at Stanford and elsewhere. The effect of the finite size of the nuclear-charge distribution in these experiments has been interpreted in terms of a form factor $f(q)$, where q is the momentum transferred to the nucleus.[1] Just what effect this would have on a process such as the bremsstrahlung arising from the scattering of such a high-energy electron by a high-Z nucleus is not immediately clear. For example, in the somewhat related process[2] of μ-pair production, finite nuclear size effect appear to reduce considerably the cross section from the result for a point nuclear-charge distribution. The smearing out of the nuclear charge might similarly be expected to reduce the bremsstrahlung cross section, especially since regions of very-high momentum transfer q are now energetically possible for BeV electrons. On the other hand, it has been argued rather qualitatively on the basis of the Bethe-Heitler formula for a point nuclear charge, that since the contributions of regions of large q to the integral for the bremsstrahlung spectrum appear to be small, the effect of finite nuclear size is not significant.[3] The higher accuracy demanded by present day experiments and some of the considerations mentioned above are the reasons for reporting the more quantitative estimates of this note.

Actually, some previous calculations of the finite nuclear size effect on electron bremsstrahlung have been made at lower energies ranging from about 20 to 100 MeV by Biel and Burhop.[4] However, it will be recalled that the characteristic angle, θ_0, for gamma emission relative to the incident electron is given by the relation $E_0\theta_0 \sim 1$, where E_0 is the incident electron energy in units of its rest energy. Unfortunately, the Biel and Burhop results extend over a range where $E_0\theta_0 \sim 10$ to 1000, so that the finite-size effects occur for correspondingly small amplitudes where, as acknowledged, they would be difficult to detect. Further, the expressions that they obtain seem to be unsuitable for the characteristic smaller angles where $E_0\theta_0$ is less than or of the order of one. In particular, although they obtain finite ratios, their expressions for the cross sections diverge and their effective minimum momentum transfer approximates to zero for forward emission. These results are not too surprising since their point source cross section agrees with an earlier calculation by Hough,[5] wherein he points out that such an expression is approximately valid for angles where $E_0\theta_0 \gg 1$. In this note, our expressions for the cross sections are not divergent for forward emission and the minimum momentum transfer is not zero. Some further comparative comments of interest are more relevently made below.

* This work has been partially supported by National Science Foundation and U. S. Army Research Office, Durham, North Carolina.

[1] For example, R. Hofstadter and R. Herman, *High-Energy Electron Scattering Tables* (Stanford University Press, Stanford, California, 1960).

[2] G. H. Rawitscher, Phys. Rev. **101**, 423 (1956). It seems appropriate to note that μ-pair production differs from bremsstrahlung in that at threshold, a relatively high minimum momentum transfer is required as compared with that associated with the electron bremsstrahlung. Thus, finite-size effects may be expected to become significant at the μ-pair threshold. However, our reason for citing this example rests on Rawitscher's Fig. 1 in which the finite-size effects appear to persist even as we go away from threshold and the minimum momentum transfer correspondingly decreases.

[3] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., p. 254.

[4] S. J. Biel and E. H. S. Burhop, Proc. Phys. Soc. (London) **68**, 165 (1955).

[5] P. V. C. Hough, Phys. Rev. **74**, 80 (1948).

II. DIFFERENTIAL CROSS SECTION

To keep the calculation of bremsstrahlung from a high-energy electron being scattered by a finite-size nucleus as simple as possible at present, we assume the validity of the Born approximation and as usual, neglect the nuclear recoil energy. The result is the familiar Bethe-Heitler expression,[6] but now modified by the multiplicative factor $|f(q)|^2$. To obtain the bremsstrahlung spectrum, this expression must be integrated over the direction of the final electron. However, because of the presence of the factor $|f(q)|^2$ which we ultimately wish to identify with the electron scattering results,[1] the integration is more conveniently reexpressed in terms of an integration over q and an appropriately related angle in a manner discussed by Rawitscher[2] but modified for this problem.

The angular part of the integration although rather long, may be done directly by contour methods and only the result is repeated here. We further note that in our calculations we are explicitly making the usual assumption that the nuclear charge distribution is spherically symmetric, so that $|f(q)|^2$ is a function of the magnitude of q only. The resulting bremsstrahlung cross section expressed somewhat more conveniently in terms of the variable $x=\frac{1}{2}q^2$ is

$$\frac{d^2\sigma}{dkd\Omega_0}=\frac{\Phi}{4\pi ka p_0}\int_{x_1}^{x_2}\frac{dx}{x^2}R(x)|F(x)|^2, \tag{1}$$

where

$$R(x)=M+Nx+AX^{-1/2}\times[x(x+n)+m+BX^{-1}(x(x+\rho)+g)],$$

$$x=\tfrac{1}{2}q^2, \qquad q=|\mathbf{p}_0-\mathbf{k}-\mathbf{p}|\equiv|\mathbf{a}-\mathbf{p}|,$$

$$x_1=\tfrac{1}{2}(a-p)^2, \qquad x_2=\tfrac{1}{2}(a+p)^2,$$

$$A=-2a/\alpha \qquad B=\tfrac{1}{2}\alpha\beta,$$

$$\alpha=E_0-p_0\cos\theta_0, \qquad \beta=\alpha-E,$$

$$r=2(1-\alpha E_0), \qquad s=\alpha p_0,$$

$$X=x(x+r)+s^2, \qquad F(x)=f(q),$$

$$M=2E^2p_0^2\sin^2\theta_0/\alpha^2-2E_0^2+4EE_0\beta/\alpha+k^2(1+k\beta/a^2),$$

$$N=1-p_0^2\sin^2\theta_0/\alpha^2+(\beta/\alpha)(k^2/a^2-2),$$

$$n=-(k^2+p_0^2+EE_0-kp_0\cos\theta_0),$$

$$m=\tfrac{1}{2}(k^2\alpha^2-4EE_0),$$

$$\rho=-[2E_0^2+\alpha(a^2-k\beta)/\beta],$$

$$g=2E_0^2\alpha(a^2-k\beta)/\beta,$$

and

$$\Phi=Z^2r_0^2/137,$$

with the notation $(E_0,\mathbf{p}_0)$ for the incident electron energy and momentum, $(E,\mathbf{p})$ for the final electron energy and momentum, $\mathbf{k}$ the photon momentum, and finally θ_0 being defined as the angle between $\mathbf{p}_0$ and $\mathbf{k}$. The units are chosen such that $\hbar=c=m_e=1$.

When evaluated, Eq. (1) will give a measure of the effect of the finite charge radius of the nucleus on the bremsstrahlung. Since the expression contains the integral of a single variable, in principle, it may be integrated numerically for any experimentally determined form factor.[1] However, some degree of caution must be advised, since our investigations, particularly that discussed below, show that, at high energies, Eq. (1) involves differences between very large and approximately equal terms.

For convenience of physical interpretation, it is desirable to evaluate the integral analytically. This can be done by choosing some suitably simple form factor. For example, we can carry through the integration for the form factor $F(x)=[1+\frac{1}{3}bx]^{-1}$ corresponding to a Yukawa charge shape. Unfortunately, such a shape is physically unrealistic for a heavy nucleus, although it can be useful to demonstrate analytic properties. However, we note that writing the cross section in terms of x clearly emphasizes the contribution of low momentum transfers through the x^{-2} factor, although the extent depends on the detailed behavior of $R(x)$ and $F(x)$. This suggests that somewhat similarly to Biel and Burhop,[4] we make the expansion

$$|F(x)|^2=1-\tfrac{2}{3}bx+\alpha_2'(bx)^2+\alpha_3'(bx)^3+\cdots, \tag{2}$$

where $b=\langle r^2\rangle$ is the mean-square-charge radius of the nucleus. The coefficients appearing in Eq. (2) are to be matched to the terms in the $x=\frac{1}{2}q^2$ expansions of more realistic form factors, say corresponding to a uniform or Woods-Saxon shaped charge distribution. In the numerical calculations of Biel and Burhop,[4] expansions up to x^4 were used.

With $|F(x)|^2$ expressed in the form of Eq. (2), we may carry out the integration in Eq. (1) to any order in x by using a standard table of integrals. Corresponding to each term in Eq. (2), we find

$$d^2\sigma/dkd\Omega_0=\sigma_{\mathrm{fs}}=\sigma_{\mathrm{ps}}+\Delta\sigma_1+\Delta\sigma_2+\Delta\sigma_3+\cdots, \tag{3}$$

where

$$4\pi kap_0\Phi^{-1}\sigma_{\mathrm{ps}}=N\ln\frac{x_2}{x_1}+A\ln\left[\frac{2(x_2+X_2^{1/2})+r}{2(x_1+X_1^{1/2})+r}\right]+A\left[\frac{mr}{2s^3}+\frac{3Bgr}{2s^5}-\frac{n}{s}-\frac{B\rho}{s^3}\right]\ln\left[\frac{x_1}{x_2}\frac{2s(X_2^{1/2}+s)+rx_2}{2s(X_1^{1/2}+s)+rx_1}\right]$$
$$+M\left[\frac{1}{x_1}-\frac{1}{x_2}\right]-\frac{A}{s^2}\left[\frac{mX_2^{1/2}}{x_2}-\frac{mX_1^{1/2}}{x_1}+\frac{Bg}{x_2X_2^{1/2}}-\frac{Bg}{x_1X_1^{1/2}}\right]+\frac{AB}{4s^2-r^2}\left[2+\frac{3gr^2}{2s^4}-\frac{4g}{s^2}-\frac{r\rho}{s^2}\right]$$
$$\times\left[\frac{2x_2+r}{X_2^{1/2}}-\frac{2x_1+r}{X_1^{1/2}}\right]+AB[3gr/2s^4-\rho/s^2][1/X_1^{1/2}-1/X_2^{1/2}], \tag{4}$$

[6] H. W. Koch and J. W. Motz, Rev. Mod. Phys. **31**, 920 (1959).

$$-4\pi kap_0\Phi^{-1}(\tfrac{2}{3}b)^{-1}\Delta\sigma_1=M\ln\frac{x_2}{x_1}+A\left(n-\tfrac{1}{2}r\right)\ln\left[\frac{2(x_2+X_2^{1/2})+r}{2(x_1+X_1^{1/2})+r}\right]+\frac{A}{s^3}[ms^2+Bg]\ln\left[\frac{x_2}{x_1}\frac{2s(X_1^{1/2}+s)+rx_1}{2s(X_2^{1/2}+s)+rx_2}\right]$$
$$+N(x_2-x_1)+A(X_2^{1/2}-X_1^{1/2})+AB\left[\frac{g}{s^2}+\frac{1}{4s^2-r^2}\left(-4s^2+2\rho r-\frac{r^2g}{s^2}\right)\right]\left[\frac{1}{X_2^{1/2}}-\frac{1}{X_1^{1/2}}\right]$$
$$+\frac{AB}{4s^2-r^2}\left[-2r+4\rho-\frac{2rg}{s^2}\right]\left[\frac{x_2}{X_2^{1/2}}-\frac{x_1}{X_1^{1/2}}\right]. \quad (5)$$

The general term in Eq. (3) involves successively simplifying integrals so that although straightforward, it is not too convenient to write explicitly. We note that σ_{ps} is the point source cross section and $\Delta\sigma_1$ represents the first shape-independent correction for finite nuclear size in the sense that it is proportional to the root-mean-square radius of the charge distribution. The explicit expressions for σ_{ps} and $\Delta\sigma_1$, will be sufficient for the purposes of this note.

III. FINITE SIZE EFFECT AT HIGH ENERGIES

Both theory and experimental results[6] confirm that for high-energy electron, where $E_0\gg1$, the bremsstrahlung is essentially confined within an angle where $\theta_0\sim1/E_0$. For the range of energies under consideration, $\theta_0^2\lesssim10^{-6}$–$10^{-8}$ so that replacing cosine and sine square by the first few terms in their series expansions is entirely reasonable. We may, thus, take advantage of the considerable simplification that occurs in our expressions for the high-energy, small-angle approximations, namely

$$E_0,\ E\gg1,\quad E_0\theta_0\sim1. \quad (6)$$

Since the limits on the integral in Eq. (1) play an important part in determining the energy dependence of our results, it is of interest to exhibit the leading order expressions explicitly. These are

$$x_2=\tfrac{1}{2}(|\mathbf{p}_0-\mathbf{k}|+p)^2\approx2E^2=2E_0^2(1-d)^2,$$
$$x_1=\tfrac{1}{2}(|\mathbf{p}_0-\mathbf{k}|-p)^2\approx k^2/[8E_0^2E^2(1+E_0^2\theta_0^2)^2]$$
$$=d^2/[8E_0^2(1-d)^2(1+E_0^2\theta_0^2)], \quad (7)$$

where the parameter $d=k/E_0$ has been introduced. Note that while x_2 increases and x_1 decreases with incident electron energy, the dependence on photon energy is vice versa for a fixed incident electron energy.

With the above approximations, we find that

$$\sigma_{ps}\approx(1-d)E_0^2\Phi2(\pi k)^{-1}$$
$$\times\left\{\frac{16E_0^2\theta_0^2}{(1+E_0^2\theta_0^2)^4}-\frac{(2-d)^2}{(1-d)(1+E_0^2\theta_0^2)^2}\right.$$
$$+\left[\frac{1+(1-d)^2}{(1-d)(1+E_0^2\theta_0^2)^2}-\frac{4E_0^2\theta_0^2}{(1+E_0^2\theta_0^2)^4}\right]$$
$$\left.\times\ln[2E_0(1-d)/d]\right\} \quad (8)$$

and

$$\Delta\sigma_1=-\tfrac{2}{3}b(1-d)E_0^2\Phi(\pi k)^{-1}$$
$$\times2\left\{\frac{3}{(1+E_0^2\theta_0^2)^2}-\frac{(2-d)^2}{2(1-d)(1+E_0^2\theta_0^2)}\right.$$
$$+\frac{1+(1-d)^2}{(1-d)(1+E_0^2\theta_0^2)}\ln[2E_0(1-d)]$$
$$\left.-\frac{2}{(1+E_0^2\theta_0^2)^2}\ln\left[\frac{2E_0(1-d)}{(1+E_0^2\theta_0^2)}\right]\right\}. \quad (9)$$

As a check of our work, we note that σ_{ps} as given in Eq. (8) corresponds to the formal given by Koch and Motz[6] under the conditions of Eq. (6).

A convenient qualitative measure of the effect of finite charge size on the bremsstrahlung cross section is given by the ratio

$$(\sigma_{fs}-\sigma_{ps})/\sigma_{ps}\approx\Delta\sigma_1/\sigma_{ps}, \quad (10)$$

For forward emission, this expression simplifies to

$$\left(\frac{\Delta\sigma}{\sigma_{ps}}\right)_{\theta_0=0}=\frac{-\tfrac{2}{3}b[d^2\ln[2E_0(1-d)]+1-d-\tfrac{1}{2}d^2]}{[1+(1-d)^2]\ln[2E_0(1-d)/d]-(2-d)^2}. \quad (11)$$

IV. RESULTS AND DISCUSSION

In Table I, we have given the results of calculating the percentage decrease in the bremsstrahlung cross section using $-\Delta\sigma_1/\sigma_{ps}$ expressed in terms of Eqs. (8) and (9) for some representative BeV electrons incident on ${}_{82}Pb^{208}$. The root-mean-square radius of this nucleus based on the Ford-Hill model[7] is taken to be $b^{1/2}=5.42\times10^{-13}$ cm which in our units corresponds to $b=1.97\times10^{-4}$.

Unlike the previous examples[4] where $E_0\theta_0\gg1$, here the finite-nuclear-size effect is quite small ranging from about 0.01% for $E_0\theta_0<\tfrac{1}{2}$, to about 1.5% for $E_0\theta_0$ near 10. For a fixed value of $E_0\theta_0$, the ratio $-\Delta\sigma_1/\sigma_{ps}$ is nearly constant over the range $E_0=0.1$ to 25 BeV. On the other hand, for a fixed angle of emission, inspection of Table I shows that $-\Delta\sigma_1/\sigma_{ps}$ increases with energy in agreement with the results of Biel and Burhop.[4] Our results also agree for the small region of values where the two calculations overlap ($E_0\theta_0=10$ at 0.1 BeV). We

[7] K. W. Ford and D. L. Hill, Ann. Rev. Nucl. Sci. 5, 25 (1955).

TABLE I. The percentage decrease, $-(100)\Delta\sigma_1/\sigma_{ps}$, in bremsstrahlung cross section for electrons of energy E_0 incident on $_{82}Pb^{208}$, ($b^{1/2}=5.42\times10^{-13}$ cm).

($d=k/E_0=0.99$)	$E_0=0.1$ BeV	1 BeV	6 BeV	25 BeV
$E_0\theta_0=$ 0	...	0.015	0.014	0.014
$E_0\theta_0=$ 0.25	...	0.016	0.015	0.015
$E_0\theta_0=$ 0.5	...	0.020	0.018	0.018
$E_0\theta_0=$ 1.0	...	0.031	0.029	0.028
$E_0\theta_0=$ 2.0	...	0.079	0.073	0.072
$E_0\theta_0=$ 4.0	...	0.264	0.247	0.242
$E_0\theta_0=$10.0	...	1.56	1.47	1.44
($d=k/E_0=0.9$)				
$E_0\theta_0=$ 0	0.014	0.012	0.012	0.012
$E_0\theta_0=$ 0.25	0.015	0.013	0.013	0.013
$E_0\theta_0=$ 0.5	0.018	0.016	0.016	0.016
$E_0\theta_0=$ 1.0	0.030	0.028	0.028	0.028
$E_0\theta_0=$ 2.0	0.090	0.079	0.075	0.071
$E_0\theta_0=$ 4.0	0.271	0.248	0.242	0.239
$E_0\theta_0=$10.0	1.59	1.46	1.42	1.41
($d=k/E_0=0.5$)				
$E_0\theta_0=$ 0	0.004	0.004	0.003	0.003
$E_0\theta_0=$ 0.25	0.005	0.005	0.005	0.005
$E_0\theta_0=$ 0.5	0.009	0.009	0.009	0.009
$E_0\theta_0=$ 1.0	0.025	0.025	0.025	0.025
$E_0\theta_0=$ 2.0	0.073	0.073	0.073	0.073
$E_0\theta_0=$ 4.0	0.240	0.238	0.237	0.237
$E_0\theta_0=$10.0	1.40	1.38	1.37	1.37

have omitted the column corresponding to $E_0=0.1$ BeV and $d=0.99$, since the high-energy approximation for the scattered electron is no longer valid for these values. It is somewhat interesting to note that because of the logarithmic dependence upon energy for the ranges of d considered, a very crude estimate to the data can be found by using the simple empirical relation

$$\Delta\sigma_1/\sigma_{ps}\approx-\tfrac{2}{3}b(1+E_0^2\theta_0^2). \quad (12)$$

Pertaining to the question of the consistency of our treatment, we note that apart from the high-energy small-angle simplifications of Eq. (6), our principal approximation was to take only the first two terms for $|F(x)|^2$ in Eq. (2). Going to the next term, after integration, we find that the leading terms give a ratio $\Delta\sigma_1/\Delta\sigma_2$ which is approximately $\alpha_2'bE_0^2(1-d)^2\sim\alpha_2'bx_2$. For the example of, say, a previously considered[4] spherical charge distribution, $\alpha_2'\approx1.9\times10^{-1}$ in our units, so neglecting second and higher order in x terms seems quite good for $d=0.99$, reasonable for $d=0.9$, and perhaps questionable for $d=0.5$. However, for this calculation, we are inclined to believe that use of a two-term shape-independent approximation for $|F(x)|^2$ may be better than comparison of successive terms may indicate. This possibility follows from the observation that all physically realistic $F(x)$ vanish as x becomes large. This is not true for $F(x)$ represented by a few terms in Eq. (2). The difference may not be important in the integral of Eq. (1) as long as x_2 is not too large, particularly since the x^{-2} factor cuts down the large x contribution. However, when x_2 becomes large, the term x^n where $n\geq2$ may not be sufficiently reduced by the x^{-2} factor to adequately duplicate the real $|F(x)|^2$ contribution to the integral in Eq. (1). Thus, at high energies, keeping only the first two terms in Eq. (2) may more nearly represent the finite size than keeping three or four. We may further speculate that the minima in the ratios at large angles and high E found in the calculations of Biel and Burhop[4] may be a consequence of representing $|F(x)|^2$ by such a truncated polynomial.

An alternative way of finding a rough quantitative estimate of the errors involved in using the first two terms in the form-factor expansion can be obtained by using a Yukawa form factor[1] which, although physically unrealisitc, does vanish at large x and can be integrated in Eq. (1). In this particular case, $|F(x)|^2$ can be written as

$$\begin{aligned}|F(x)|^2&=[1+(\tfrac{1}{3}b)x]^{-2}\\&=1-2(\tfrac{1}{3}b)x+2(\tfrac{1}{3}b)^2x^2[1+(\tfrac{1}{3}b)x]^{-1}\\&\qquad+(\tfrac{1}{3}b)^2x^2[1+(\tfrac{1}{3}b)x]^{-2}. \quad (13)\end{aligned}$$

If we take b to be the mean square radius used in our calculations of Table I, the last two terms represent an exact correction due to terms neglected in our treatment of $|F(x)|^2$. When these terms are integrated in Eq. (1) and evaluated according to the approximations of Eq. (6), we find that the effect of using an exact Yukawa form factor as opposed to the two-term shape-independent expansion is to approximately reduce the values obtained in Table I by less than 25%.

In conclusion, we find that for the characteristic angles of emission, the effect of nuclear size on the bremsstrahlung spectrum is quite small for electrons in the BeV-energy range, crudely following Eq. (12) for the range of energies and d considered. For more accurate results using experimentally determined form factors, numerical integration of Eq. (1) is available. For the values considered in this note, nuclear recoil energy corrections are not significant, however, as a practical matter, the neglected role of inelastic effects as a correction to the usual bremsstrahlung formula should also be considered, as well as the more usual electron screening effects.

We wish to gratefully acknowledge the discussions with members of the Cambridge Electron Accelerator staff who stimulated our interest in this problem.